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Citation

José Madrid, 'Sharp inequalities for the variation of the discrete maximal function', *Bulletin of the Australian Mathematical Society* **95**(1) (2017), 94–107.

This is a paper about the regularity properties of the discrete analogues of the classical Hardy–Littlewood maximal function. The question of how maximal functions act on Sobolev spaces is natural and well studied. The author considers a version of this question in the discrete setting and proves a quantitative result with best constant. The examples showing optimality are simple and natural and the proofs are clever but not overly technical. Moreover, the paper points out an intriguing phenomenon: the proof given in dimension 1 extends to higher dimensions when intervals are replaced by ℓ_1 balls, but not when they are replaced by ℓ_∞ balls. A different proof is given in the latter case, showing the complexity of the higher dimensional problem through its dependence on choices of summation methods. Given that the continuous case is only fully understood in dimension 1, the paper may give further insight on its difficulty. To give a flavour of the results, the sharp inequality in dimension 1 arises as follows. For $f : \mathbb{Z} \mapsto \mathbb{R}$ in $\ell_1(\mathbb{Z})$, define its centred Hardy–Littlewood maximal function

$$Mf(n) = \sup_{r \in \mathbb{Z}^+} \frac{1}{(2r+1)} \sum_{k=-r}^{r} |f(n+k)|.$$

Then $\mathrm{Var} Mf \leq 2\|f\|_{\ell_1 \mathbb{Z}}$. This result solved a problem posed by J. Bober, E. Carneiro, K. Hughes and L. B. Pierce, 'On a discrete version of Tanaka's theorem for maximal functions', *Proc. Amer. Math. Soc.* **140** (2012), 1669–1680.