The George Szekeres Medal
2020

Ole Warnaar

Ole Warnaar, Chair and Professor of Pure Mathematics, University of Queensland, is a leading expert in special functions, partition theory and algebraic combinatorics. He made a number of breakthrough contributions in each of these fields, often making use of techniques from a variety of different areas of mathematics to solve outstanding problems. He and his collaborators were the first to extend the celebrated Rogers–Ramanujan identities to infinite families of affine Kac–Moody algebras. He developed a beautiful theory of partial theta functions, was one of the pioneers in the field of elliptic hypergeometric functions, and is one of the leading experts on the Selberg integral and its applications.

Ole is a Fellow of the Australian Academy of Science, has held senior positions in AustMS, and is its current incoming President-Elect. He contributes further to the Australian mathematical sciences community through his work on the ARC College of Experts, and the MATRIX and AMSI Scientific Boards. He is responsible for The University of Melbourne/ACEMS School Mathematics Competition. A major international role of service and leadership was as one of four managing editors over many years of the *Journal of Combinatorial Theory, Series A*, the leading classical and algebraic combinatorics journal; he remains one of five advising editors.

Extended citation

Ole Warnaar is a leading expert in special functions, partition theory and algebraic combinatorics, who has made a number of significant breakthroughs, some of which now bear his name. Ole often uses techniques from different areas of mathematics to solve outstanding problems. He regularly publishes in the top journals in pure mathematics and is widely acknowledged by his peers as a leader in his field. Ole has also contributed significantly to academic life within the Australian mathematical sciences community. He was elected at a relatively young age to Fellowship of the Australian Academy of Science and throughout his career he has held several senior positions in the Australian Mathematical Society, and is its current incoming President-Elect. Ole is an outstanding supervisor and lecturer, an accomplished mathematician and a worthy recipient of the George Szekeres Medal of the Australian Mathematical Society.

Six particular highlights of Ole's work are described below.

The Rogers–Ramanujan identities are two of the most beautiful and intriguing q-series identities, and have played an important role in partition theory, analytic number theory, statistical mechanics, representation theory and conformal field theory. Since they were first discovered by Rogers in 1894, many generalisations have been found by, among others, Andrews, Bailey, Dyson, Gordon, Selberg and Slater. From a representation theoretic point of view, essentially all of these arise in the representation theory of the affine Lie algebra $A_1^{(1)}$ (or affine $\mathfrak{sl}_2$). With his coauthors, Ole developed several entirely new approaches to Rogers–Ramanujan identities based on symmetric function theory (Hall–Littlewood polynomials, Kostka polynomials, Macdonald–Koornwinder theory) and the representation theory of W and infinite dimensional Lie algebras to prove infinite hierarchies of Rogers–Ramanujan identities for most infinite families of affine Kac–Moody Lie algebras. The three main publications covering this work have all appeared (or are set to appear) in journals of the highest calibre: *Memoirs Amer. Math. Soc.*, *Duke Math. J.*, *J. Amer. Math. Soc.*. The *Duke* paper extensively generalises previous arithmetic studies of Rogers–Ramanujan identities and was featured in *Discover Magazine*.

The Selberg integral (1944) is widely regarded as one of the most important hypergeometric integral evaluations, because of its intrinsic beauty and the key role it played in so many seemingly disparate areas of mathematics. It plays a central role in random matrix theory, describes (at least conjecturally) the statistical properties of the Riemann zeros (leading to a conjectural closed-form formula for the even moments of the Riemann zeta-function), allows for the computation of correlation functions in conformal field theory, and describes the solutions of the Knizhnik–Zamolodchikov differential equations for the Lie algebra $\mathfrak{sl}_2$.

Using Macdonald polynomials theory, Ole succeeded, more than 60 years after Selberg to generalise the Selberg integral from $\mathfrak{sl}_2$ to $\mathfrak{sl}_n$. This confirmed the $\mathfrak{sl}_n$ case of a deep conjecture of Mukhin and Varchenko, who proposed that, if the space of singular vectors arising from the tensor product of two $\mathfrak{g}$-Verma modules is one-dimensional, then there should be a corresponding type-$\mathfrak{g}$ Selberg integral evaluation. Recently, the $\mathfrak{sl}_n$ Selberg integral was applied in relation to the famous AGT conjecture (Alday, Gaiottom Tachikawa, 2010) which predicts a link between between $N = 2$ superconformal field theory in four dimensions and Liouville conformal field theory on a punctured Riemann surface. The paper describing the Selberg integral was published in *Acta Math.*, one of the top journals in pure mathematics; it was followed by a paper with with Peter Forrester in the *Bulletin Amer. Math. Soc.* which has been cited over 250 times.

In his *Lost Notebook*, Ramanujan stated, without proof, a number of beautiful identities for what are now known as partial theta functions. Using q-series methods, Ole managed to provide a unifying framework for Ramanujan's partial theta function identities and to embed each such identity into an infinite family. The key breakthrough in this work was the realisation that the famous Jacobi triple product identity for ordinary theta functions admits a generalisation to partial theta functions. His paper on partial theta functions published in *Proc. London Math. Soc.* resulted in the chapter entitled "The Warnaar Theory" in Andrews and Berndt's edited version of Ramanujan's *Lost Notebook*. Quotes from that chapter are "Warnaar has developed an extended and beautiful theory of partial theta functions" and "Warnaar's striking generalisation of the Jacobi triple product identity."

"Elliptic special functions" are a relatively new area of research in the theory of special functions, which grew out of the remarkable observation by Frenkel and Turaev in 1997 that one of the most important results in the theory of basic hypergeometric functions has a modular analogue. Ole, in the foundational paper "Summation and transformation formulas for elliptic hypergeometric series", was the first to systematically study elliptic hypergeometric series, both in the one-variable and in the root system setting. The paper introduces what are now known as "Warnaar's elliptic determinant" and "Warnaar's elliptic matrix inversion" and has been the basis for a new chapter on "Elliptic, modular and theta hypergeometric series" in the second edition of Gasper and Rahman's classic text "Basic Hypergeometric Series", and an invitation by the editors of the Askey–Bateman project to contribute two chapters on elliptic hypergeometric functions.

In a subsequent paper appearing in *Advances Math.*, he and coauthor V. P. Spiridonov introduced what is now often referred to as the elliptic Fourier transform. This transform has played an important role in Seiberg-type dualities in four-dimensional supersymmetric quantum field theories and in the elliptic double-affine Hecke algebra. In the paper "The superconformal index of the E_6 SCFT", Gadde et al. resolve the Argyres–Seiberg duality with E_6 symmetry and write in their introduction "Technically, this is possible thanks to a remarkable inversion formula for a class of integral transforms", referring to Ole's *Advances* paper.

The Kostka number $K_{\lambda\mu}$ gives the dimension of the weight-space indexed by μ in the irreducible $\mathrm{GL}(n,\mathbb{C})$-module indexed by λ. It has a natural q-analogue which can be used to compute the characters of the general linear groups over finite fields as well as the Poincaré polynomials of partial unipotent flag varieties. In 1974 Foulkes famously asked for a combinatorial explanation of the positivity of the Kostka–Foulkes polynomials in terms of a yet-to-be found statistic on semi-standard Young tableaux. Several years later this was, equally famously, answered by Lascoux–Schützenberger who introduced the notions of cyclage and the charge of a Young tableaux. Ole (with A. Schilling) used ideas from Kashiwara's theory of crystal bases and Yang–Baxter integrable systems to extend the Lascoux–Schützenberger theory from Young tableaux to Littlewood–Richardson tableaux. *Math. Reviews* described the resulting publication in *Comm. Math. Phys.*, as a "landmark paper".

Two-dimensional exactly solvable lattice models in statistical mechanics (or $(1+1)$-dimensional solvable quantum spin chains) provide important toy models for testing notions such as universality and predictions arising from conformal field theory. One of the most famous of all solvable lattice models is the Ising model, first solved in two dimensions in 1944 by Nobel Laureate Lars Onsager. In a paper in *Phys. Rev. Lett.* (with B. Nienhuis and K. Seaton), Ole discovered what are now known as the dilute ADE lattice models. These models are significant in that they provide the first-ever examples of solvable lattice models in the presence of a symmetry breaking field. Importantly, the dilute A_3 model is in the same universality class as the critical Ising model in a magnetic field, a model that itself has remained famously unsolved. With V. V. Bazhanov and B. Nienhuis he also showed in a paper in *Phys. Lett. B* that the spectrum of this model can be described in terms of the exceptional Lie algebra E_8, confirming a prediction by Zamolodchikov. In 2010 this E_8 structure was observed experimentally for the first time in one-dimensional chains of $\mathrm{CoNb_2O_6}$.