Systems of isometries

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AustMS meeting, Monash University, December 2019

I wish to acknowledge the people of the Kulin Nations, on whose land we are gathered today. I pay my respects to their Elders, past and present.

Overview

1. Group algebras in Algebra and in C^*-algebra
2. Common themes in C^*-algebras
3. C^*-algebras of semigroups, in particular, quasi-lattice ordered groups

Let G be a discrete group and let k be a field.
In Algebra, the group algebra of G over k is the set

$$k[G] := \{f : G \to k \colon f \text{ has finite support}\},$$

with $+$ defined pointwise and $*$ defined by

$$(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x).$$

Notice that $i : G \to k[G]$ where $i(x) = \delta_x$.

- Universal property of $k[G]$: for any group homomorphism $\pi : G \to A^\times$ into the set of invertible elements of a k-algebra A, there is a unique homomorphism $\tilde{\pi} : k[G] \to A$ such that $\pi = \tilde{\pi} \circ i$.
- Any algebra satisfying this property is isomorphic to $k[G]$.

In C^*-algebra, we take $k = \mathbb{C}$, we complete $\mathbb{C}[G]$ in a norm and use representations by unitary operators on a Hilbert space.

Let H be a Hilbert space with norm $\|h\| = (h \mid h)^{1/2}$. Let $B(H)$ be the algebra of bounded linear operators $T : H \to H$.

Examples

- $H = \mathbb{C}^n$, $B(H) \cong M_n(\mathbb{C})$.
- $H = \ell^2(\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$ of square-summable sequences in $\mathbb{C}$.
- $H = \ell^2(G) = \{f : G \to \mathbb{C} : \sum_{x\in G} |f(x)|^2 < \infty\}$.

For each $T \in B(H)$ there exists a unique $T^* \in B(H)$, called the adjoint of T such that $(Th \mid k) = (h \mid T^* k)$ for all $h, k \in H$.

- $T \mapsto T^*$ is the $*$ in C^*-algebra.
- $B(H)$ is the prototypical C^*-algebra.
- Abstract C^*-algebras A have a $*$ and are complete in a norm such that $\|a^* a\| = \|a\|^2$ for $a \in A$.

$U \in B(H)$ is unitary if $UU^* = 1 = U^* U$, that is, $U^* = U^{-1}$.

Completing $\mathbb{C}[G]$ to a C^*-algebra

Let $U : G \to B(H)$ be a unitary representation: a homomorphism such that each $U(x)$ is unitary.
Example: The left-regular representation $\lambda : G \to B(\ell^2(G))$ where $(\lambda_y f)(x) = f(y^{-1}x)$ for $f \in \ell^2(G)$ is unitary.
Define $\pi_U : \mathbb{C}[G] \to B(H)$ by

$$\pi_U(f) = \sum_{x \in G} f(x)U(x).$$

Then π_U is a representation, and $\|\pi_U(f)\| \leq \sum_{x \in G} |f(x)| < \infty$.
There is a norm on $\mathbb{C}[G]$ defined by

$$\|f\| := \sup\{\|\pi_U(f)\| : U \text{ is a unitary representation of } G\}.$$

The group C^*-algebra $C^*(G)$ is the completion of $\mathbb{C}[G]$ in this norm.
Universal property: $C^*(G)$ is generated by a unitary repn i such that for every unitary repn $U : G \to B(H)$, there is a unique repn $\pi_U : C^*(G) \to B(H)$ such that $U = \pi_U \circ i$, and every repn of $C^*(G)$ arises in this way.

The reduced group C^*-algebra is defined using the left-regular representation:

$$C_r^*(G) := \pi_\lambda(C^*(G)) = \langle \lambda_x : x \in G \rangle \subseteq B(\ell^2(G)).$$

Obvious question: When is $\pi_\lambda : C^*(G) \to C_r^*(G)$ an isomorphism?
Answer: If and only if G is an amenable group (Hulanicki, 1966).

Difficult question: When is $C_r^*(G)$ simple? If and only if the action of G on the Furstenberg boundary is topologically free (Kalantar–Kennedy, 2017).

For Thompson's groups F and T, $C_r^*(T)$ is simple if and only if F is not amenable (Haagerup–Olesen, 2017 and Bleak–Juschenko, 2018).

Common themes

- We build a C^*-algebra from a combinatorial object to get a universal and a reduced (concrete) C^*-algebra;
- we study when the two are the same, leading to notions of amenability.
- The properties of the C^*-algebras reflect the properties of the object - so these are effective ways to build examples.
- We obtain classes to test hypotheses arising, for example, in the classification program for C^*-algebras (eg. nuclearity).

There are several different ways of norming an algebraic tensor product $A \odot B$ of C^*-algebras and completing it to a C^*-algebra. Say A is nuclear if for all B there is only one C^*-norm on $A \odot B$. (Takesaki, 1964)

Question: When is $C^*(G)$ nuclear? Answer: If and only if G is an amenable group.

C^*-algebras of semigroups

What are the appropriate representations?

$S \in B(H)$ is an isometry if $\|Sh\| = \|h\|$ for all $h \in H$. This happens iff $S^*S = 1$. Often S is nonunitary: $SS^* \neq 1$.

Example: the unilateral shift

$H = \ell^2(\mathbb{N})$, $S((x_0, x_1, x_2, \dots)) = (0, x_0, x_1, x_2, \dots)$. Then $S^*((y_n)) = (y_1, y_2, y_3, \dots)$, and $S^*S = 1$ and $SS^* \neq 1$.

The C^*-algebra generated by an isometry $S \in B(H)$ is:

$$C^*(S) := \overline{\text{span}}\{\text{all products of } S, S^*\}$$
$$= \overline{\text{span}}\{S^m (S^*)^n : m, n \in \mathbb{N}\}.$$

Coburn's Theorem, 1967

Let a be an isometry in an abstract C^*-algebra A such that $aa^* \neq 1$, and let S be the unilateral shift on $H = \ell^2(\mathbb{N})$. Then there is an isomorphism $\pi : C^*(a) \to C^*(S)$ such that $\pi(a) = S$.

Quasi-lattice ordered groups

Let P be a generating submonoid of a discrete group G st $P \cap P^{-1} = \{e\}$. For $x, y \in G$ define

$$x \leq y \iff x^{-1}y \in P \iff y \in xP.$$

Then $\leq$ is a partial order on G with $PP^{-1} = \{x \in G : x \text{ has upper bounds in } P\}$.

Definitions

(G, P) is lattice ordered if any $x, y \in G$ have a least common upper bound in P. (Equivalently, any $x \in G$ has a least upper bound in P.)

(G, P) is quasi-lattice ordered if any $x, y \in G$ with a common upper bound in P have a least common upper bound $x \vee y$ in P (Nica, 1992).

(G, P) is weakly quasi-lattice ordered if any $x, y \in P$ with a common upper bound in P have a least common upper bound in P.

Lemma

(G, P) is lattice ordered if and only if $G = PP^{-1}$ and (G, P) is quasi-lattice ordered.

Examples

1. $(\mathbb{Z}^2, \mathbb{N}^2)$ under $+$. Here $(m_1, m_2) \leq (n_1, n_2)$ iff $m_i \leq n_i$. So $(m_1, m_2) \vee (n_1, n_2) = (\max(m_1, n_1, 0), \max(m_2, n_2, 0))$.
2. Let G be the free group on $\{a, b\}$, and P the submonoid consisting of words in a and b. Then $x, y \in P$ and $x \leq y$ means that x is an initial segment of y, and the rest of y has no factors of a^{-1} or b^{-1}. Here $x \vee y = \infty$ often.

Motivating example

Let $0 \neq c, d \in \mathbb{Z}$. Consider the Baumslag–Solitar group:

$$G = \langle a, b : ab^c = b^d a \rangle.$$

Let P be the submonoid generated by a and b.

Lemma (Spielberg, 2012)

If $c, d \in \mathbb{N}^\times$, then (G, P) is quasi-lattice ordered. If c, d have opposite signs, then (G, P) is weakly quasi-lattice ordered.

For simplicity, assume that $c, d \in \mathbb{N}^\times$. Crucial in the proof is that G is an HNN extension of $\mathbb{Z}$, and hence each $x \in G$ has a unique normal form.

Write $\theta : G \to \mathbb{Z}$ for the height map, the homomorphism such that $\theta(a) = 1$ and $\theta(b) = 0$. If $x \in P$ and $k = \theta(x)$, then

$$x = b^{s_0} a b^{s_1} \cdots b^{s_{k-1}} a b^{s_k}.$$

where each $0 \leq s_0, \ldots, s_{k-1} < d$ and $s_k \geq 0$.

- $a^2b^{42} = b^0ab^0ab^{42}$
- $b^da = ab^c = b^0ab^c$
- If $0 \leq n < d$, then $b^{n+d}a = b^nab^c$.

Spielberg's proof that the B–S group is quasi-lattice ordered uses that (G, P) is quasi-lattice ordered iff (Crisp-Laca, 2002):

> if $x \in PP^{-1}$, then there exist a pair $\alpha, \beta \in P$ with $x = \alpha\beta^{-1}$ such that $\gamma, \delta \in P$ and $\gamma\delta^{-1} = \alpha\beta^{-1}$ imply $\alpha \leq \gamma$ and $\beta \leq \delta$. (The pair α, β is unique.)

Example: Thompson's group

$$F = \left\langle \{x_i : i \in \mathbb{N}\} : x_i x_k = x_k x_{i+1} \text{ for } i > k \geq 0 \right\rangle.$$

Let P be the submonoid generated by $\{x_i : i \in \mathbb{N}\}$. Then (F, P) is quasi-lattice ordered (Nucinkis, 2017), and since $F = PP^{-1}$ is lattice ordered. This follows from a normal form for elements of F.

Idea of normal form

If $\alpha \in P$, write it uniquely as a product of increasing generators $\alpha = x_{n_1}^{\epsilon_1} \ldots x_{n_j}^{\epsilon_j}$ where $n_1 < \cdots < n_j$. If $x \in F$, the relation $x_k^{-1} x_i = x_{i+1} x_k^{-1}$ implies that $x \in PP^{-1}$. Then we look for gaps and reduce further: Example:

$$x_3 x_5 x_7 (x_4 x_5 x_9 x_{11})^{-1} = x_3 x_6 x_5 (x_4 x_8 x_{10} x_5)^{-1} = x_3 x_6 (x_4 x_8 x_{10})^{-1}.$$

The normal form for $x \in F$ is given by the unique pair $\alpha, \beta \in P$ such that $x = \alpha\beta^{-1} \in PP^{-1}$ and $x = \gamma\delta^{-1} \in PP^{-1}$ implies $\alpha \leq \gamma$ and $\beta \leq \delta$.

C^*-algs of a quasi-lattice ordered grp

Take $\ell^2(P)$ with orthonormal basis $\{e_x : x \in P\}$ the point masses. For each $x \in P$, there is an isometry T_x on $\ell^2(P)$ such that $T_x e_y = e_{xy}$ for $y \in P$. Then T is an isometric repn of P on $\ell^2(P)$ called the Toeplitz repn. The Toeplitz algebra is the C^*-algebra generated by the Toeplitz repn, that is,

$$\mathcal{T}C^*(P) := \langle\{T_x : x \in P\}\rangle \subseteq B(\ell^2(P)).$$

Nica observed that

$$T_x^* T_y = \begin{cases} T_{x^{-1}(x\vee y)} T^*_{y^{-1}(x\vee y)} & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty, \end{cases} \tag{1}$$

and we call isometric repns satisfying (1) Nica covariant. It follows that $\mathcal{T}C^*(P) = \overline{\operatorname{span}}\{T_x T_y^* : x, y \in P\}$.

There is a C^*-algebra $C^*(P)$ which is universal for Nica covariant representations.

Say (G, P) is amenable if $\pi_T : C^*(P) \to \mathcal{T}C^*(P)$ is an isomorphism.
Questions: When is $C^*(P)$ nuclear and when is (G, P) amenable?
Answer: If (G, P) is weakly quasi-lattice ordered, then $C^*(P)$ nuclear implies (G, P) is amenable (X. Li, 2017). We have some sufficient conditions for nuclearity and/or amenability.

Propn/Examples (Nica, 1992)

- If G is an amenable group, then (G, P) is amenable.
- Let $\mathbb{F}_2$ be the free group on $\{a, b\}$ and let $\mathbb{F}_2^+ = \langle a, b\rangle$. Then $(\mathbb{F}_2, \mathbb{F}_2^+)$ is amenable.

Tease

If (G, P) is lattice ordered and an amenable quasi-lattice ordered group, then G is an amenable group (Crisp-Laca).

- So Thompson's group F is amenable as a group if and only if (F, P) is amenable as a quasi-lattice ordered group.

Theorem (Clark-aH-Raeburn, 2016)

The Baumslag-Solitar (G, P) is amenable.

To prove this we had to generalise the proof of:

Generalised length fns

Theorem (Laca-Raeburn, 1996)

Let (G, P) and (K, Q) be quasi-lattice ordered groups and let $\theta : G \to K$ be "controlled". If K is an amenable group, then (G, P) is an amenable quasi-lattice ordered group.

Controlled means: θ is an order-preserving group homomorphism such that for $x, y \in P$ with $x \vee y < \infty$:

1. $\theta(x) \vee \theta(y) = \theta(x \vee y)$;
2. $\theta(x) = \theta(y) \Rightarrow x = y$.

Example

Let $\mathbb{F}_2$ be the free group on $\{a, b\}$. Let $\theta\colon \mathbb{F}_2 \to \mathbb{Z}$ be the homomorphism such that $\theta(a) = 1 = \theta(b)$. Then $\theta\colon (\mathbb{F}, \mathbb{F}^+) \to (\mathbb{Z}, \mathbb{N})$ is a controlled map.

Example: Baumslag-Solitar $G = \langle a, b : ab^c = b^d a \rangle$

The height map where $\theta(a) = 1$, $\theta(b) = 0$ is not controlled because $e \vee b^n < \infty$ and $\theta(e) = 0 = \theta(b^n)$ for $n \in \mathbb{N}$.

Ideas from the proof that the Baumslag–Solitar (G, P) is amenable led to a new notion of "controlled map".

New "controlled" map (aH–Raeburn–Tolich, 2018)

Let (G, P) and (K, Q) be weakly quasi-lattice ordered groups, and $\theta : G \to K$ be an order-preserving group homomorphism st

1. For $x, y \in P$ with $x \vee y < \infty$, have $\theta(x) \vee \theta(y) = \theta(x \vee y)$.
2. Let $q \in Q$. Set $\Sigma_q = \{\sigma \in \theta^{-1}(q) \cap P : \sigma \text{ is minimal}\}$.
   a. If $x \in \theta^{-1}(q) \cap P$, then there exists $\sigma \in \Sigma_q$ such that $\sigma \leq x$.
   b. If $\sigma, \tau \in \Sigma_q$, then $\sigma \vee \tau < \infty \Rightarrow \sigma = \tau$.

Example: Baumslag–Solitar $G = \langle a, b : ab^c = b^d a \rangle$

Let $\theta : (G, P) \to (\mathbb{Z}, \mathbb{N})$ be $\theta(a) = 1$, $\theta(b) = 0$. Here

$$\theta^{-1}(q) \cap P = \{b^{s_0} a b^{s_1} \cdots b^{s_{q-1}} a b^{s_q} : 0 \leq s_0, \ldots, s_{q-1} < d, s_q \in \mathbb{N}\}$$
$$\Sigma_q = \{b^{s_0} a b^{s_1} \cdots b^{s_{q-1}} a : 0 \leq s_0, \ldots, s_{q-1} < d\},$$

and θ is a controlled map.

Theorem (aH–Nucinkis–Sehnem–Yang, 2019)

Suppose that $\theta : (G, P) \to (K, Q)$ is controlled (in a new technical sense) and that K is an amenable group. If $C^*(\ker\theta \cap P)$ is nuclear, then $C^*(P)$ is nuclear.

Very rough proof idea:

Use θ to construct a coaction $\delta_\theta : C^*(P) \to C^*(P) \otimes_{\min} C^*(K)$ and study the structure of the "fixed-point algebra"

$$\{a \in C^*(P) \colon \delta_\theta(a) = a \otimes e\} = \overline{\operatorname{span}}\{i_x i_y^* : \theta(x) = \theta(y)\}$$

of the coaction. The amenability of K means that certain properties of the fixed-point algebra transfer to $C^*(P)$ (for example, nuclearity!).

Example: bootstrapping using the theorem

Let $G = \langle a, b : ab^c = b^d a \rangle$. The height map $\theta : (G, P) \to (\mathbb{Z}, \mathbb{N})$ is controlled. Then $\ker\theta \cap P = \{b^t : t \in \mathbb{N}\} \cong \mathbb{N}$ and $C^*(\mathbb{N})$ is nuclear. So $C^*(P)$ is nuclear and (G, P) is amenable.

Example (tease)

Let F be the Thompson group and $\theta : (F, P) \to (\mathbb{Z}, \mathbb{N})$ be the homomorphism such that $\theta(x_0) = 1$ and $\theta(x_i) = 0$ for $i \geq 1$. Here θ is a controlled map! But.... $\ker \theta \cap P = \langle x_1, x_2, \dots \rangle \cong P$.

We have a more general notion of controlled map which deals with the infinite descending chains appearing in the Baumslag–Solitar group where c, d have opposite signs.

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