# Large deviations in random graphs

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(joint work with Matan Harel, Gady Kozma, and Frank Mousset)

The 63rd Annual Meeting of the Australian Mathematical Society
Melbourne, December 6, 2019

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**Question**

How 'concentrated' is $X_N$ around its expectation?

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Two possible ways to strengthen this result:

- How fast can $\varepsilon$ tend to zero as $N \to \infty$?
- What is the rate of convergence?

# Typical deviations – Central Limit Theorem

The standard deviation $\sigma$ of $Y_1$ is defined by

$$\sigma := \sqrt{\mathsf{Var}(Y_1)} = \left(\mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]^2\right)^{1/2}.$$

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The limiting behaviour depends only on $\mathbb{E}[Y_1]$ and $\mathbb{E}[Y_1^2]$.

# Large deviations – Cramér's theorem

## Theorem (Cramér 1938)

There is a function $I = I_{Y_1} \colon (0, \infty) \to (0, \infty]$ such that

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as $Y_1, \ldots, Y_N$ are i.i.d. We choose the optimal value of $\lambda$ (...) □

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A natural example coming from random graph theory:

$$X_N = \#\text{triangles in } G_{n,p};$$

here, $N = \binom{n}{2}$ and $X_N$ may be expressed as degree-three polynomial in $N$ independent Bernoulli random variables.

# Triangles in the binomial random graph

The binomial random graph $G_{n,p}$ has vertex set $\llbracket n \rrbracket := \{1, \ldots, n\}$ and

$$\Pr\left(ij \in G_{n,p}\right) = p \quad \text{for all } i, j \in \llbracket n \rrbracket,$$

independently of all other pairs.

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Let $X_N$ denote the number of triangles in $G_{n,p}$ and note that

$$X_N = \sum_{i,j,k} Y_{ij} Y_{ik} Y_{jk} \qquad \text{and} \qquad \mathbb{E}[X_N] = \binom{n}{3} p^3,$$

where $Y_{ij} = \mathbf{1}_{ij \in G_{n,p}} \sim \text{Bernoulli}(p)$.

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**Remark**

We will allow $p$ to depend on $n$. In fact, assume $p = p(n) \to 0$ as $n \to \infty$.

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**Theorem (Ruciński 1988)**

If $p \gg 1/n$, then, for every fixed $x \geqslant 0$,

$$\lim_{N\to\infty} \Pr\left(|X_N - \mathbb{E}[X_N]| \geqslant x \cdot \sigma_N\right) = F(x) := \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2}\, du,$$

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The standard deviation of $X_N$ is straightforward to compute:

$$\sigma_N^2 = \mathrm{Var}(X_N) = \binom{n}{3} p^3(1-p^3) + \binom{n}{4}\binom{4}{2} p^5(1-p).$$

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## Problem (lower tail)

For a given $\delta \in (0, 1]$, detemine the asymptotics of

$$\Pr\left(X \leqslant (1-\delta)\mathbb{E}[X]\right).$$

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We conclude that

$$\Pr\big(X \geqslant (1+\delta)\mathbb{E}[X]\big) \geqslant \exp\big(-c_\delta n^2 p^2 \log(1/p)\big).$$

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The assumption $p \gg \log n/n$ is necessary.

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**Question**

Can we compute $\log \Pr\big(X \geqslant (1+\delta)\mathbb{E}[X]\big)$ asymptotically?

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Can we compute $\log \Pr\big(X \geqslant (1+\delta)\mathbb{E}[X]\big)$ asymptotically? (What for?)

Define

$$\psi(\delta) = \min\big\{e(G) : \mathbb{E}[X \mid G \subseteq G_{n,p}] \geqslant (1+\delta)\mathbb{E}[X]\big\}.$$

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If $\psi(\delta) \to \infty$, then

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**Theorem (Lubetzky–Zhao 2014)**

$$\psi(\delta)/n^2p^2 \to \begin{cases} \delta^{2/3}/2 & \text{if } n^{-1} \ll p \ll n^{-1/2}, \\ \min\{\delta^{2/3}/2, \delta/3\} & \text{if } n^{-1/2} \ll p \ll 1. \end{cases}$$

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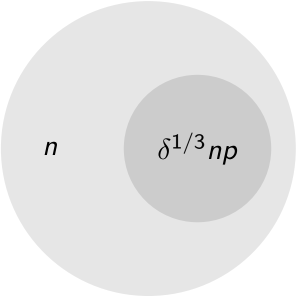


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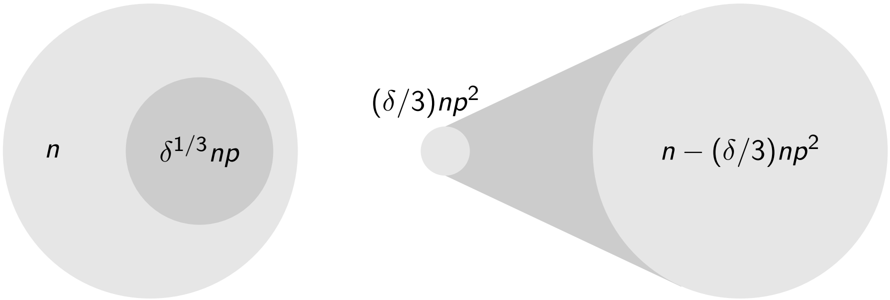


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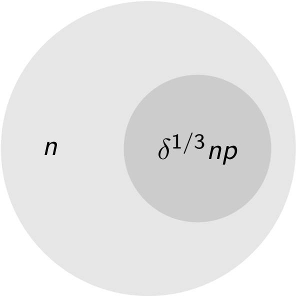


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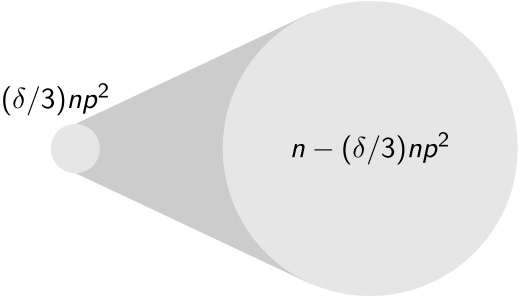


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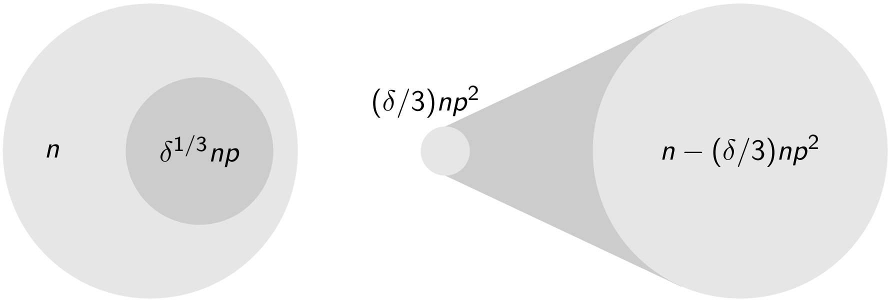


clique containing $\delta\mathbb{E}[X]$ triangles

$K_3$-free 'hub' with $\delta\mathbb{E}[X]/p$ copies of $K_{1,2}$ (each becomes a triangle with probability $p$)

The 'hub' works only when $np^2 \gg 1$, as $(\delta/3)np^2$ is assumed an integer.

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We expect the following to be true (the assumption $p \ll 1$ is needed):

## Theorem

If $n^{-\alpha} \ll p \ll 1$, then, for every $\delta > 0$,

$$\Pr\left(X \geqslant (1+\delta)\mathbb{E}[X]\right) \leqslant p^{(1-o(1))\cdot\psi(\delta)}.$$

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## Theorem (Harel–Mousset–S. 2019)

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**Theorem (Harel–Mousset–S. 2019)**

If $1/n \ll p \ll \log n/n$, then, for every $\delta > 0$,

$$\Pr\left(X \geqslant (1+\delta)\mathbb{E}[X]\right) = \exp\left(-(1+o(1))\mathrm{Po}(\delta)\cdot\mathbb{E}[X]\right).$$

where $\mathrm{Po}(\delta) = (1+\delta)\log(1+\delta) - \delta$.

# Lower tail

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## Corollary (Harris 1960 / Janson 1990)

If $p < .99$, then, for every $\delta \in (0,1]$,

$$\Pr\big(X \leqslant (1-\delta)\mathbb{E}[X]\big) = \exp\big(-\Theta_\delta\big(\min\{n^2 p, n^3 p\}\big)\big).$$

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## Theorem (Łuczak 2000)

If $p \gg n^{-1/2}$, then $\Pr(X = 0) \leqslant (1-p)^{n^2/4 - o(n^2)}$.

# Lower tail (revisited)

If $\delta < 1$, then we could consider a graph $G_\delta$ with at most $(1-\delta)\binom{n}{3}$ triangles and as many edges as possible to obtain

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Choose $q\colon \binom{[\![n]\!]}{2} \to [0,1]$ and let $G_{n,q}$ be the random graph on $[\![n]\!]$ s.t.:

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## Proposition

Suppose that $q$ is such that $\mathbb{E}\left[\#K_3(G_{n,q})\right] \leqslant (1-\delta)\mathbb{E}[X] = (1-\delta)\binom{n}{3}p^3$. Then,

$$\Pr\left(X \leqslant (1-\delta)\mathbb{E}[X]\right) \geqslant \exp\left(-(1+o(1)) \cdot \sum_{i,j} I_p(q_{ij})\right),$$

where $I_p(q) = q \log \frac{q}{p} + (1-q)\log\frac{1-q}{1-p}$.

# Our contribution

Define, for every $\delta \in (0,1]$,

$$\Phi(\delta) = \min \left\{ \sum_{i,j} I_p(q_{ij}) : \mathbb{E}[\# K_3(G_{n,q})] \leqslant (1-\delta)\mathbb{E}[X] \right\}.$$

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We have

$$\frac{\Phi(1)}{\log(1-p)} = \mathrm{ex}(n, K_3) - \binom{n}{2} = \left\lfloor \frac{n^2}{4} \right\rfloor - \binom{n}{2},$$

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**Theorem (Kozma–S. 2019+)**

If $n^{-1/2} \ll p \leqslant .99$, then, for every $\delta \in (0,1]$,

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**Thank you for your attention!**