On total variation flow type equations

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1. Introduction

What is the total variation flow equation?

$$\frac{\partial u}{\partial t} - \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

It is interpreted as the L^2 gradient flow of the total variation energy

$$E_1(u) = \int_\Omega |\nabla u|\, dx,$$

where Ω is a domain in $\mathbf{R}^N$ or a Riemannian manifold.

L^2 gradient flow of p-Dirichlet energy (p-diffusion equation)

Gradient flow $u_t = -\delta E/\delta u$ of the p-Dirichlet energy $E = E_p$

$$E_p(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx$$

is

$$u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

If $p = 2$, it is the heat equation:

$$u_t - \Delta u \equiv 0.$$

If $p = 1$, it is the total variation flow.

Structure of the p-diffusion equation

$$u_t - \mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0$$

- $2 < p < \infty$: parabolicity (diffusion effect) is degenerate at $\mathrm{grad}\, u = 0$.
- $1 < p < 2$: parabolicity is strong (singular) at $\mathrm{grad}\, u = 0$.
- $p = 1$: total variation flow
  parabolicity is singular at $\mathrm{grad}\, u = 0$.
  parabolicity is degenerate in the direction of $\mathrm{grad}\, u$.

One dimensional example

$$u_t - \partial_x(\operatorname{sgn} u_x) = 0$$

$$\operatorname{sgn}\sigma = \begin{cases} 1, & \sigma \geq 0 \\ -1, & \sigma < 0. \end{cases}$$

Formally,

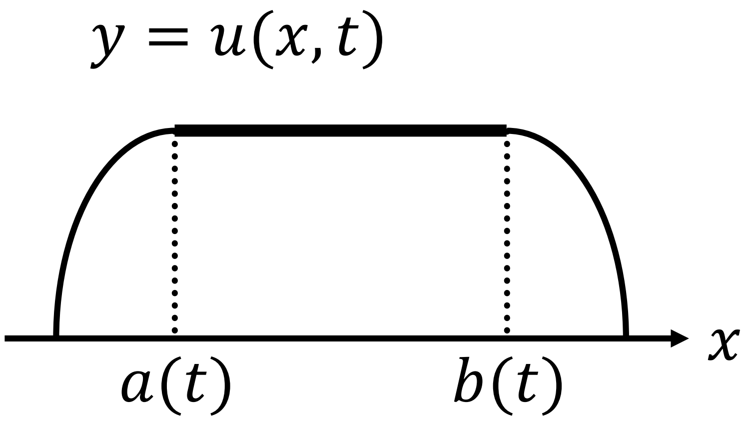
$$u_t - 2\delta(u_x)u_{xx} = 0.$$

- Diffusion effect is strong at $u_x = 0$ while it is zero at $u_x \neq 0$. The definition of solution is nontrivial.

Speed of the place where $u_x = 0$

Question. How does the profile in the figure move?

Ansatz: Flat part (facet) does not split nor bend so that u_t is constant on the facet.



Integrate $u_t = \partial_x(\operatorname{sgn} u_x)$ from $a - \varepsilon$ to $b + \varepsilon$ to get

$$\int_{a-\varepsilon}^{b+\varepsilon} (\operatorname{sgn} u_x)_x \, dx = \operatorname{sgn} u_x\,(b+\varepsilon) - \operatorname{sgn} u_x\,(a-\varepsilon)$$
$$= -1 - (+1)$$

provided $\varepsilon > 0$ is sufficiently small.

Nonlocality of the speed

By Ansatz u_t is constant on (a, b) so the $LHS \to u_t(b-a)$ as $\varepsilon \to 0$. Thus

$$u_t(b-a) = -1 - (+1),$$

i.e.,

$$u_t = \frac{-2}{b-a} = \frac{\#\text{ end point}}{\text{facet length}}.$$

This is a nonlocal quantity called **Cheeger's ratio**.

- Does Ansatz follow from approximation?

  yes: $N = 1$, no: $N \geq 2$

2. Various examples

From the theory of crystal growth

(1) $u_t - \sqrt{1 + |\nabla u|^2}\, \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$ (non-divergence type)

More generally, crystalline mean curvature flow equation

(J. Taylor 1991, S. B. Angenent – M. E. Gurtin 1991)

(2) $u_t + \Delta\left[\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \alpha\, \mathrm{div}(|\nabla u|\nabla u)\right] = 0,\ \alpha \geq 0$

(H. Spohn 1993)

Example similar to $u_t - \mathrm{div}(\nabla u/|\nabla u|) = 0$ but having a weaker singularity:

$$u_t - |\nabla u|\, \mathrm{div}(\nabla u/|\nabla u|) = 0.$$

(level-set mean curvature flow equation)

(Y.-G. Chen – YG – S. Goto 1991, L. C. Evans – J. Spruck 1991)

From image denoising

(3) 1-harmonic map flow equation

$$u_t = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| u$$

$u: \Omega \to S^2 =$ two dimensional unit sphere in $\mathbf{R}^3$

Removing noise from chromaticity. It is the gradient flow of total variation with manifold constraint of S^2.

(B. Tang – G. Sapiro – V. Caselles 2000)

3. Variational approach

If the equation has the divergence structure, the theory of maximal monotone operators works well.

Advantage: It applies to higher order equations including example (2).

Disadvantage: It needs divergence structure so does not apply (1) and (3).

A general theory of maximal monotone operators

Lemma 1 (Y. Kōmura 1967, H. Brezis – A. Pazy 1970). Let H be a real Hilbert space. Let $\mathcal{E} : H \to \mathbf{R} \cup \{+\infty\}$ be convex, lower semicontinuous, $\mathcal{E} \not\equiv \infty$. Then, there exists a unique solution $u \in C([0,\infty), H) \cap \mathrm{Lip}((\delta,\infty), H)$ $({}^\forall \delta > 0)$ of

$$\frac{du}{dt} \in -\partial\mathcal{E}(u) \quad \text{a.e.} \quad t > 0, \quad u(0) = u_0.$$

Moreover, u is right-differentiable for $t > 0$ and

$$\frac{d^+u}{dt} = -\partial^0\mathcal{E}(u), \quad t > 0$$

Solution knows how to evolve! Note $\overline{D(\partial\mathcal{E})} = \overline{D(\mathcal{E})}$.

Subdifferential and minimal section

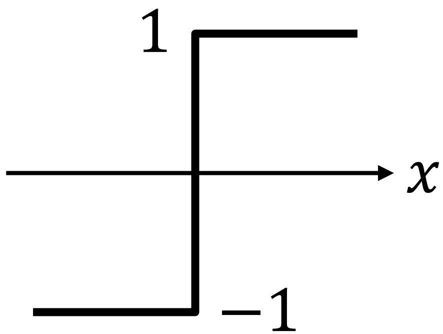
$\partial\mathcal{E}(u) := \{\, f \in H \mid \mathcal{E}(u+v) - \mathcal{E}(u) \geq \langle v, f\rangle_H \text{ for all } v \in H\}$

$\partial\mathcal{E}$: **subdifferential** operator

(Its element is called a **subgradient**.)

If $\mathcal{E}$ is differentiable at u, $\partial\mathcal{E}(u)$ is a singleton.

- Example: $\mathcal{E}(x) = |x|, \quad x \in \mathbf{R} \Rightarrow \partial\mathcal{E}(x) = \begin{cases} \{1\} & x > 0 \\ [-1,1] & x = 0 \\ \{-1\} & x < 0 \end{cases}$



- Note that $\partial\mathcal{E}$ is a maximal monotone operator if $\mathcal{E}$ is convex and lower semicontinuous.

$$\partial^0\mathcal{E}(u) := \operatorname{argmin}\{\|f\|_H \mid f \in \partial\mathcal{E}(u)\}$$

(**minimal section**)

Precise definition of total variation energy under periodic boundary conditions

We set $H = L^2(\mathbf{T}^N)$ and

$$E(u) = \begin{cases} \displaystyle\int_{\mathbf{T}^N} |\nabla u|, & u \in BV(\mathbf{T}^N) \cap H \\ \infty, & u \in H \setminus BV. \end{cases}$$

Note that E is lower semicontinuous in H. It is also lower semicontinuous in another Hilbert space which is less regular.

$$H_{\mathrm{av}}^{-1}(\mathbf{T}^N) = H_{\mathrm{av}}^1(\mathbf{T}^N)^*, H_{\mathrm{av}}^1(\mathbf{T}^N) = \left\{ f \in H^1(\mathbf{T}^N) \middle| \int_{\mathbf{T}^N} f\, dx = 0 \right\}$$

$$\langle f, g \rangle_{H_{\mathrm{av}}^1} = \int_{\mathbf{T}^N} \nabla f \cdot \nabla g \, dx$$

Unique solvability

Second order: $u_t - \mathrm{div}(\nabla u / |\nabla u|) = 0$

$$H = L^2(\mathbf{T}^N), \quad \mathcal{E} = E$$

Fourth order: $u_t + \Delta\, \mathrm{div}(\nabla u / |\nabla u|) = 0$

$$H = H^{-1}_{\mathrm{av}}(\mathbf{T}^N), \quad \mathcal{E} = E$$

Equation: $\partial_t u \in -\partial_H E(u)$

We apply Lemma 1 to get unique solvability.

Proposition 2. There exists a unique global-in-time solution for $\partial_t u \in -\partial_H E(u)$ with initial data $u_0 \in H(L^2(\mathbf{T}^N)$ or $H^{-1}_{\mathrm{av}}(\mathbf{T}^N))$.

Fourth order problem: R. V. Kohn – YG 2012

Fractional total variation flow: D. Hauer – J. M. Mazón 2019

Subdifferentials

- In general, it is difficult to calculate subdifferentials. For total variation energy with respect to L^2, it is well studied. (F. Andreu-Vaillo – V. Caselles – J. M. Mazón “Parabolic Quasilinear Equations Minimizing Linear Growth Functionals” 2004)
- If $u : \mathbf{T}^N \to \mathbf{R}$ takes minimum value m and $U = \{x \in \mathbf{T}^N | u(x) = m\}$ (facet) has a smooth boundary, then

$$\partial E(u)\Big|_U = \left\{ -\operatorname{div} z \in L^2 \;\middle|\; \begin{array}{r} |z| \le 1 \text{ in } U \\ z \cdot \nu = 1 \text{ on } \partial U \end{array} \right\},$$

where ν is the outer unit normal provided that $u \in D(\partial E)$.

Minimal section

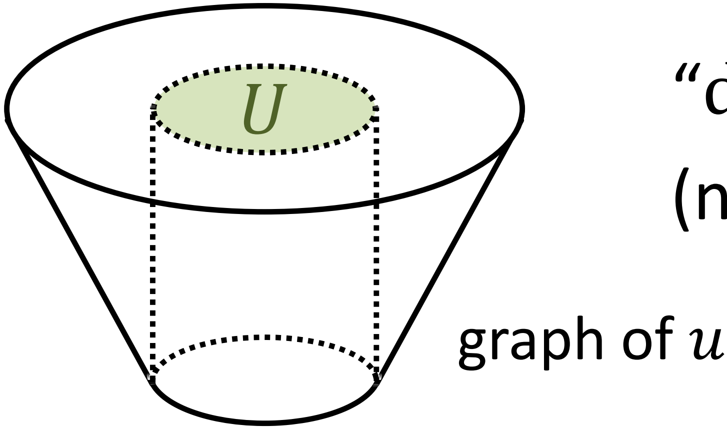
The value $\partial^0 E(u)$ at U is obtained by solving the obstacle problem:

(P) $$\min\left\{ \int_U |\operatorname{div} z|^2 \, dx \;\middle|\; \begin{array}{l} |z| \le 1 \ \text{ in } \ U \ (\text{constraint}) \\ z \cdot \nu = 1 \ \text{ on } \ \partial U \end{array} \right\}.$$

If z_0 is a minimizer, $\partial^0 E(u)|_U = -\operatorname{div} z_0$ (uniquely determined).

"$\operatorname{div} z_0 = \text{const.} \Leftrightarrow$ \`Ansatz'"

(no facet splitting nor bending)



Calibrability

Definition 3. A set U is **calibrable** if there exists a vector field z such that $\operatorname{div} z = \text{const}$ in U satisfying the constraint $|z| \leq 1$ in U and the boundary condition $z \cdot \nu = 1$ or ∂U

- For such z, its divergence $\operatorname{div} z$ is the minimizer of (P).
- One dimensional interval is calibrable.
- Any ball is calibrable.
- Ellipsoid may not be calibrable if eccentricity is large.
- Various characterization is now available (G. Bellettini – V. Casseles – M. Novaga 2002 … S. Amato – G. Bellettini – L. Tealdi 2015).

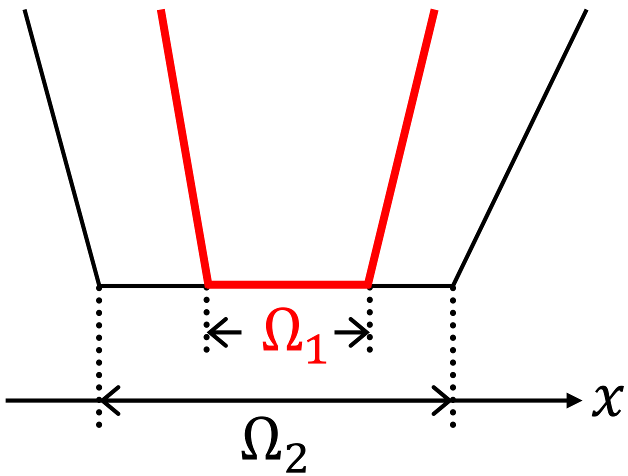
Nonlocal quantity playing a role of second derivatives

Definition 4. Let z_0 be a minimizer of (P). We set $\Lambda(U) :=$ div z_0 and call **nonlocal curvature** (of the graph of u).

Theorem 5 (special case of M.-H. Giga – YG – N. Požár 2014).

$$\Omega_1 \subset \Omega_2 \;\Rightarrow \Lambda(\Omega_2) \leq \Lambda(\Omega_1) \;\text{ a.e. on }\; \Omega_1$$

(cf. YG – M. Gurtin – J. Matias 1996)



Idea of proof: Consider the resolvent approximation

$$(I + a\partial E)(v) \ni u, \quad a > 0.$$

Let v_a be the solution. Then $-\partial^0 E(u) = \lim_{a \downarrow 0}(v_a - u)/a$. Comparison of Λ is reduced to maximum principle of the resolvent equation.

Relation to Cheeger's ratio

If U is calibrable, then

$$\Lambda(U) = |\partial U|/|U| \quad (\textbf{Cheeger's ratio})$$

$$\because \quad \int_U \operatorname{div} z_0 = \int_{\partial U} z_0 \cdot \nu := |\partial U|,$$

$$LHS = \Lambda(U) \cdot |U|.$$

4. Viscosity approach

Theory based on maximum principle

Advantage: No divergence structure is necessary.

Disadvantage: Applies only to second-order problem.

Theorem 6 (M.-H. Giga – YG – N. Požár 2014). For any $u_0 \in C(\mathbf{T}^N)$, there exists a unique global-in-time solution $u \in C\left(\mathbf{T}^N \times [0,\infty)\right)$ satisfying

$$u_t + f\left(\nabla u, \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) = 0, \quad t > 0$$

with initial data u_0. Here f is continuous and non increasing in the last variable (so that the problem is degenerate parabolic).

Example: $u_t - \sqrt{1 + |\nabla u|^2}\, \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$

Definition of viscosity solutions

- Standard definition for elliptic problem

  A function $u \in C(D)$ is a viscosity subsolution of

$$F(x, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad D$$

  if $(\varphi, \hat{x}) \in C^2(D) \times D$ satisfies

$$F(\hat{x}, \nabla\varphi(\hat{x}), \nabla^2\varphi(\hat{x})) \leq 0$$

  whenever $\max_D (u - \varphi) = (u - \varphi)(\hat{x})$.

  supersolution: replace $\leq$ by $\geq$, $\max$ by $\min$

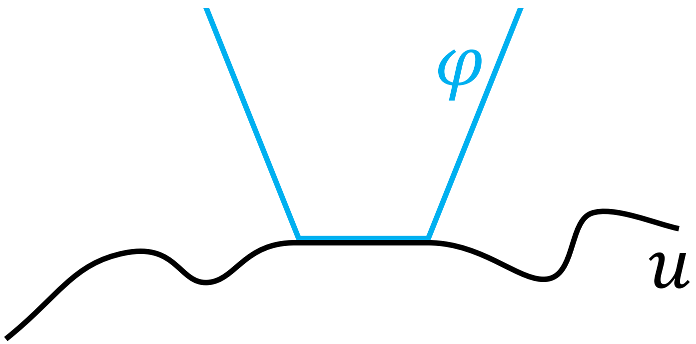
Idea of definition of viscosity subsolution for our problem

Define suitable class of test function φ so that nonlocal curvature for facets is defined. $\Lambda[\varphi] = \mathrm{div}(\nabla\varphi/|\nabla\varphi|)$ should be defined.

A function u is a viscosity **subsolution** to $u_t + f(\nabla u, \mathrm{div}(\nabla\varphi/|\nabla\varphi|)) = 0$ in Q if $(\varphi, (\hat{x}, \hat{t}))$ satisfies

$$\varphi_t(\hat{x}, \hat{t}) + f\big(\nabla u, (\hat{x}, \hat{t}), \Lambda[\varphi](\hat{x}, \hat{t})\big) \le 0$$

whenever $\max_Q (u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})$ where $Q = \boldsymbol{T}^N \times (0, T)$.



Comparison principle

Theorem 7. If u is a subsolution and v is a supersolution, then $u \leq v$ at $t = 0$ implies $u \leq v$ in Q.

Idea of the proof (involved)

(1) Doubling variable

(2) Flattening argument (M.-H. Giga, YG '01, 1D)
(to compare values of $\Lambda(\Omega)$ easily)

(3) Comparison of nonlocal curvature
(Approximate general facets by facets with smoother boundary so that the nonlocal curvature is well-defined.)

5. Crystalline mean curvature flow

5.1 Mean curvature flow

Let $\{\Gamma_t\}_{t\geq 0}$ be an evolving hypersurface in $\mathbf{R}^N$. The **mean curvature flow equation** is an equation for $\{\Gamma_t\}$ of the form

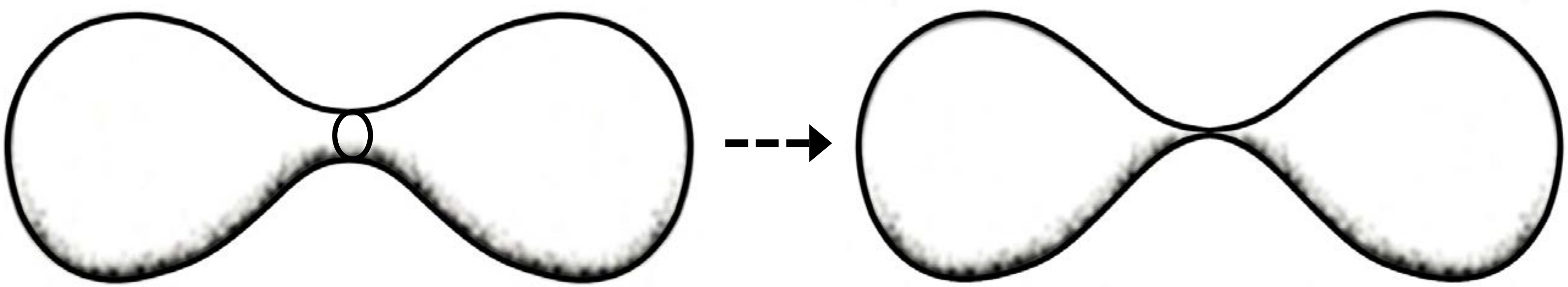
$$V = H \text{ on } \Gamma_t.$$

Here V is the normal velocity and H is ($N-1$ times) mean curvature.

Source: Materials Science [W. W. Mullins 1956] (motion of grain boundaries in annealing metals)

Formation of singularities

initial data | pinching



Γ_0 | Γ_t

dumbbell with thin neck | [M. Grayson 1989]

Weak solutions including singularity

- Variational approach: K. Brakke 1972
  T. Ilmanen 1993, … Y. Tonegawa and et al 2014.
- A level set method: Y.-G. Chen – YG – S. Goto 1991
  L. C. Evans – J. Spruck 1991

  [Book: YG, Surface evolution equations, Birkhäuser 2006]
  (encounter with the theory of viscosity solution, A deterministic game interpretation, R. V. Kohn – S. Serfaty 2006 …)

Extension to anisotropic flow

Is it possible to extend a level set approach to anisotropic curvature flow?

Yes, Y.-G. Chen – YG – S. Goto 1991
provided that the interfacial energy density is convex and **smooth**.

What happens the interface energy density is not C^1 (strong anisotropy)?

5.2 Crystalline mean curvature flow

A **crystalline mean curvature flow** is a typical example of **anisotropic** mean curvature flow, which for example describes motion of antiphase grain boundaries.

Anisotropic mean curvature is a change ratio of an interfacial energy with respect to variation of volume enclosed by a hypersurface Γ in $\mathbf{R}^N$.

Interfacial energy

Let γ_0 be a nonnegative continuous function defined on a unit sphere S^{N-1}, which is called an **interfacial energy density**. For a given hypersurface Γ we set

$$I(\Gamma) = \int_{\Gamma} \gamma_0(\vec{n})\, d\mathcal{H}^{N-1}$$

which is called an **interfacial energy**. Here $\vec{n}$ denotes the unit exterior normal of Γ and $d\mathcal{H}^{N-1}$ is the surface element.

Anisotropic mean curvature

The anisotropic mean curvature H_γ is defined by

$$H_\gamma = -\frac{\delta}{\delta\Gamma} I(\Gamma).$$

It is explicitly written as

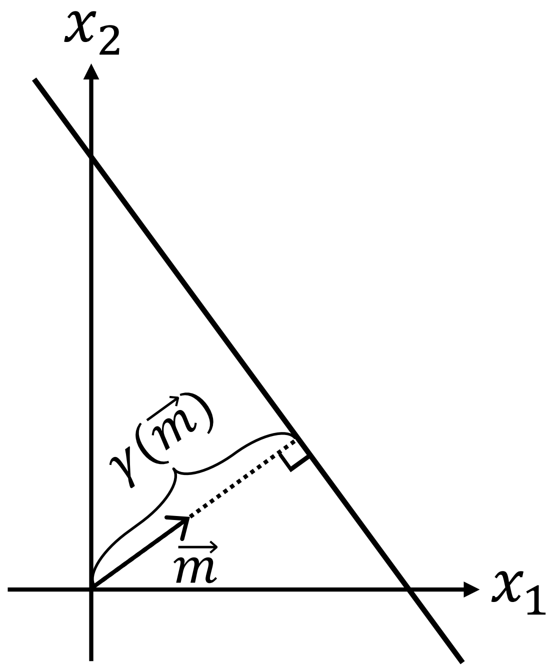
$$H_\gamma = -\mathrm{div}_\Gamma\, \xi\,(\vec{n}) \quad \text{on} \quad \Gamma,$$

where $\xi\,(p) = \nabla_p \gamma(p)$ and γ is the homogenization of γ_0 i.e., $\gamma(p) = \gamma_0(p/|p|)|p|$. Here div_Γ denotes the surface divergence. The vector field $\xi(\vec{n})$ is often called the Cahn-Hoffman vector field.

Wulff shape
— a substitute for the sphere

$$W_\gamma = \bigcap_{|\vec{m}|=1} \{\, x \in \mathbf{R}^N \,|\, x \cdot \vec{m} \le \gamma(\vec{m}) \}$$

$$\Rightarrow \; H_\gamma = -(n-1) \quad \text{on} \quad \Gamma = \partial W_\gamma$$



for smooth γ. [The converse is true provided that Γ is compact and embedded and that γ_0 is smooth and "strictly convex". Anisotropic version of Alexandrov's theorem. (Y. He – H. Li – H. Ma – J. Ge 2009)]

Crystalline mean curvature

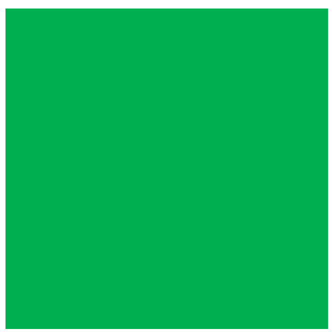
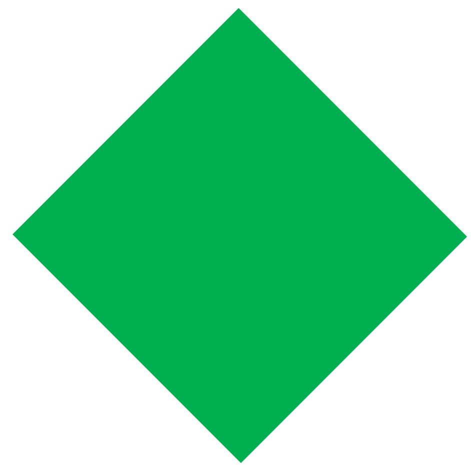
If $\gamma_0 \equiv 1$, then $I(\Gamma)$ is nothing but the area of Γ and H_γ is $(N-1 \text{ times})$ the mean curvature. In this case $\gamma(p) = |p|$. In general, γ may not be convex nor smooth.

We say that γ_0 (or γ) is **crystalline** if γ is convex and piecewise linear. An anisotropic mean curvature H_γ is a **crystalline mean curvature** if γ is crystalline.

Wulff shape for a crystalline energy

$$F_\gamma = \{\, p \in \mathbf{R}^N \mid \gamma(p) \le 1 \,\}$$ Frank diagram

($W_\gamma =$ polar of F_γ)

F_γ	W_γ
	
regular polyhedron	its dual

Anisotropic mean curvature flow

Let V denote the normal velocity of an evolving (hyper)surface Γ_t. A general form of anisotropic mean curvature flow equation is

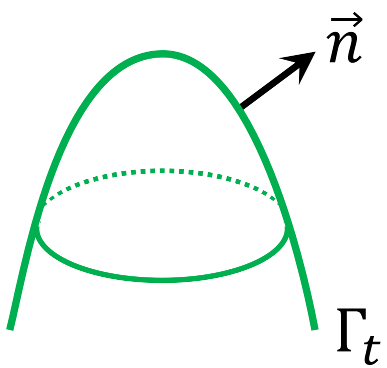
$$V = f\left(\vec{n}, H_\gamma\right) \quad \text{on} \quad \Gamma_t , \qquad \text{(ACF)}$$

where f is a given function. Example includes

(i) $V = H_\gamma$

(ii) $V = M(\vec{n})\left(H_\gamma + C\right),\ C \in \mathbf{R}$,

where $M(\vec{n}) > 0$ is a given function.



Crystalline mean curvature flow

We say that (ACF) is a **crystalline mean curvature flow equation** if f is continuous and non-decreasing in H_γ and H_γ is a crystalline mean curvature. It is formally a degenerate parabolic equation of the second order. However, as we see later it is a very singular equation.

More examples: $V = |H_\gamma|^{\alpha-1} H_\gamma, \quad \alpha > 0,$

$$V = 1 - \exp(-H_\gamma).$$

We find faceted shapes in various crystals

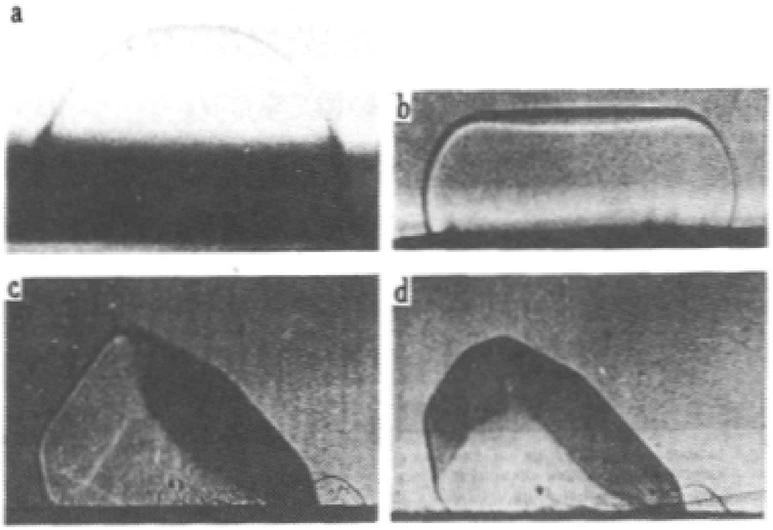


図 4.3.2 ^{4}He の結晶の形．温度はそれぞれ，(a) 1.3 K，(b) 1.08 K，(c) 0.4 K，(d) 0.35 K．(S. Balibar, F. Gallet, and R. Rolley : J. Cryst. Growth 99 (1990) 46 より)

^{4}He crystal at near absolute zero temperature. If the temperature is lower, we observe more flat portion called facet.

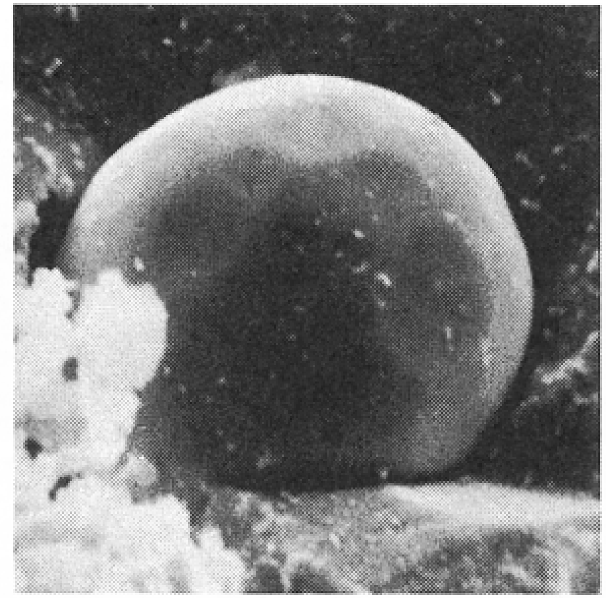
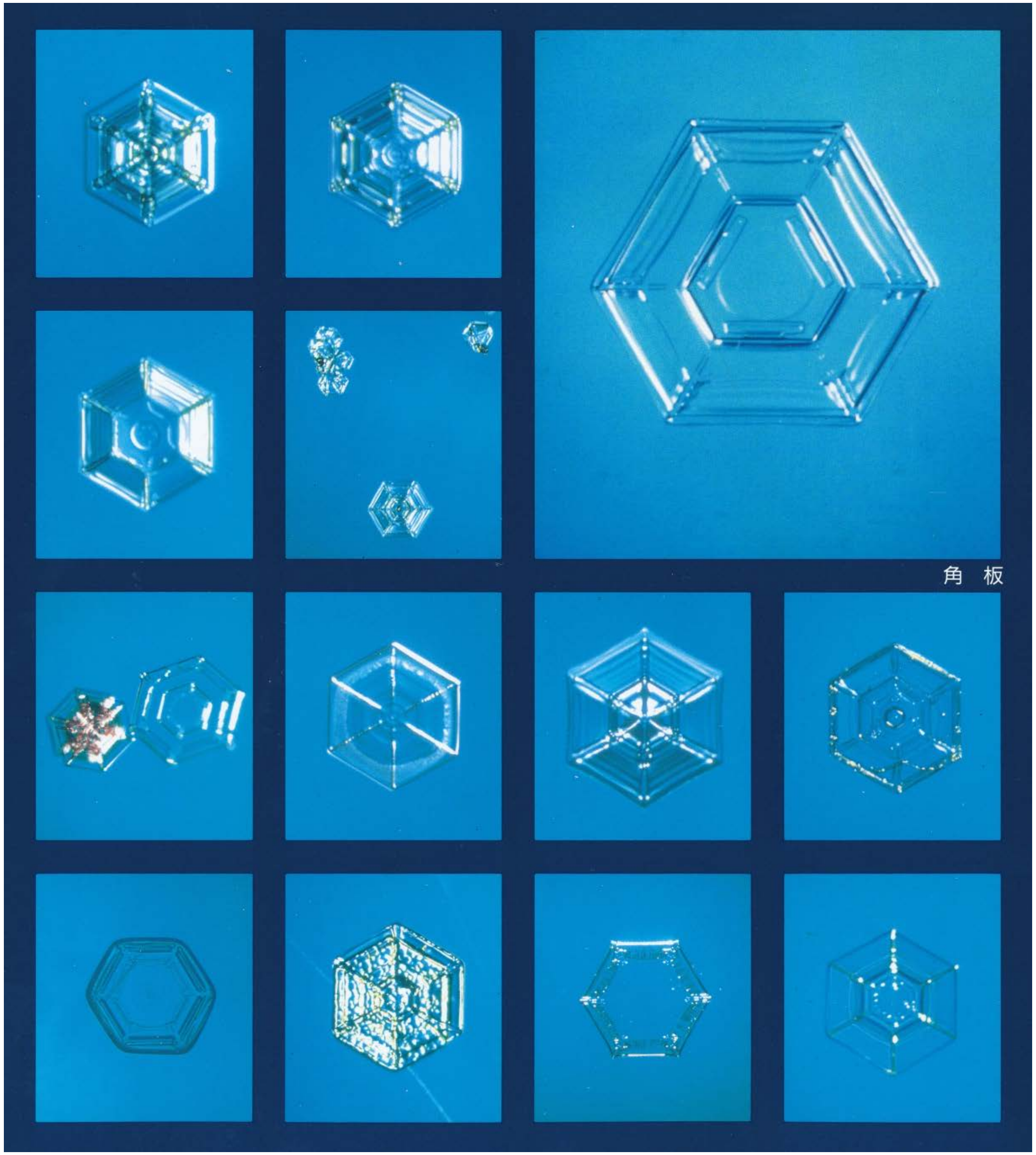


図 9.1.13 固体金属銀と共存状態でのセレン化銀 α-Ag_2Se の平衡形．{100}，{110} 面以外の高次の面が観察されている．

In a crystal of silver selenide a lot of facets appear.

photos from
“Handbook of crystal growth”,
Kyoritsu (1995) (in Japanese)



Teisaku Kobayashi, Yoshinori Furukawa, Snow crystals, Snow Crystal Museum, Asahikawa, Japan (1991)

5.3 Problems

Problem A (Existence and Uniqueness). Consider the crystalline mean curvature flow equation, for example,

(i) $$V = H_\gamma \quad \text{on} \quad \Gamma_t.$$

Does the initial value problem admit a unique solution in $\mathbf{R}^N$?

[For a given closed surface $\Gamma_0 \subset \mathbf{R}^N$ are there a unique family $\{\Gamma_t\}_{t\geq 0}$ solving (i)?]

Problem B (Stability). Is this solution (if exists) approximable by smoothed anisotropic mean curvature flow?

[If interfacial energy γ is approximable by γ_ε, does the corresponding solution $\{\Gamma_t^\varepsilon\}_{t\geq 0}$ approximate Γ_t?]

5.4 Known results

- Well-studied for planar motion

  S. B. Angenent – M. Gurtin 1989, J. Taylor 1991

  Level set method: M.-H. Giga – YG 2001 ARMA

- Higher dimension

  Even local existence was not known unless initial data is convex.

  A unique existence of a global flow provided that initial data is **convex**.

  G. Bellettini – V. Caselles – A. Chambolle – M. Novaga 2006 ARMA

5.5 Level set flow

This is a **generalized solution** of curvature flow equations which allows topological change of the flow Γ_t. We consider a level-set equation of (ACF). A level-set of the solution is regarded as a level-set flow.

Example

For the mean curvature flow equation, its level-set equation is of the form

$$v_t - |\nabla v| \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = 0.$$

We consider this equation in $\mathbf{R}^N \times (0, \infty)$ not only on Γ_t.

- The initial value problem is uniquely solvable in the sense of viscosity solutions with uniformly continuous initial data.
- $\Gamma_t = \{x | v(x,t) = 0\}$ is uniquely determined by Γ_0.
- $D_t = \{x | v(x,t) > 0\}$ is uniquely determined by D_0.

cf. S. Osher – J. Sethian 1989 (numerics), Y. G. Chen – YG – S. Goto 1991, L. C. Evans – J. Spruck 1991, … YG Surface Evolution Equations 2006

5.6 Main Results

Theorem 8 (Existence and Uniqueness, YG – N. Požár 2016, 2018). For a given initial data (a bounded open set) $D_0 \subset \mathbf{R}^N$ with its boundary Γ_0 there exists a global level-set flow D for **crystalline** mean curvature flow equation (ACF).

Level set equation for $V = f\left(\vec{n}, H_\gamma\right)$ on Γ_t:

$$v_t - |\nabla v|\ f\left(-\frac{\nabla v}{|\nabla v|}, -\mathrm{div}\left(\nabla\gamma(-\nabla v)\right)\right) = 0.$$

Theorem 9 (Stability, YG – N. Požár 2016). Assume the same hypotheses of Theorem 8. Let γ_ε be a smooth convex interfacial energy density approximate γ uniformly. Then the level-set flow D_ε converges to D in the Hausdorff distance sense in space-time provided that no fattening occurs.

(Underlying structure: comparison principle)

(YG – N. Požár, Adv. Differential Equations, 2016, $N = 3$)

(YG – N. Požár, CPAM, 2018, N: arbitrary)

Recent related work by A. Chambolle, M. Morini and M. Ponsiglione (2017, 2019)

Problem: $V = \gamma H_\gamma$ on $\mathbf{R}^N$

Result: For any open set D_0 in $\mathbf{R}^N$ there exists a unique level-set like flow.

Merits: No assumptions on γ other than convexity
No boundedness for D_0
No restriction on space dimensions

Very recent: + M. Novaga extends to

$$V = \beta\left(\kappa_\gamma + f\right),$$

β: convex mobility $\beta = \beta(\vec{n})$, $f = f(x,t)$ given.

Further comparison and a key idea

Disadvantage: The equation can be handled is "linear" in κ_γ with convex coefficient.

Idea: If D is a (super) solution, then one expects that the anisotropic distance d fulfills the equation

$$d_t - \operatorname{div} z \geq 0, \; z \in \partial\gamma(\nabla d)$$

outside D.

Comparison principle

We have to define a correct notion of a viscosity solution and conclude that if initially $u \leq v$ for sub and super solution, then $u \leq v$ for all time. For this purpose, we define **anisotropic** nonlocal curvature $\Lambda(\Omega)$ to compare. We also need approximation by "smoother" facets.

Remarks

- Viscosity theory has been established for $N = 2$ for a long time ago (M.-H. Giga – YG 2001).
- The minimal divergence is constant for curve evolution but may not be constant for surface evolution.
- A breakthrough is done for total variation flow of non-divergence type (M.-H. Giga – YG – N. Požár 2014).

$$u_t - \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\,\frac{\nabla u}{|\nabla u|} = 0.$$

- Extension to inhomogeneous driving force $V = f\left(\vec{n}, H_\gamma + g(x)\right)$ (YG – N. Požár, work in progress)

Remarks (behavior of solutions)

- Evolution of polygonal curve by crystalline flow

  M.-H. Giga – YG 2000

  B. Andrews 2002 convex curve

  T. Ishiwata 2008 almost convex phenomena

- Self-similar shrinking solutions of higher genus

  N. Požár 2019

Summary

We give several approaches for total variation flow type equations.

We also apply our viscosity approaches to crystalline mean curvature flow to get global solvability.

Some open problems

- If the equation is of fourth order and has no divergence structure well-posedness is left open.

  Example: $u_t = \Delta \exp(-\operatorname{div}(\nabla u / |\nabla u|))$

- For one-harmonic map flow, there are many results for existence but the uniqueness is widely left open.